

A note on compactness properties of the singular Toda system

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Abstract

In this note, we consider blow-up for solutions of the $SU(3)$ Toda system on a compact surface Σ . In particular, we give a complete proof of the compactness result stated by Jost, Lin and Wang in [11] and we extend it to the case of singularities. This is a necessary tool to find solutions through variational methods.

1 Introduction

Let (Σ, g) be a smooth, compact Riemannian surface. We consider the $SU(3)$ Toda system on Σ :

$$-\Delta u_i = \sum_{j=1}^2 a_{ij} \rho_j \left(\frac{V_j e^{u_j}}{\int_{\Sigma} V_j e^{u_j} dv_g} - \frac{1}{|\Sigma|} \right) - 4\pi \sum_{j=1}^l \alpha_{ij} \left(\delta_{p_j} - \frac{1}{|\Sigma|} \right) \quad i = 1, 2 \quad (1)$$

with $\rho_i > 0$, $0 < V_i \in C^\infty(\Sigma)$, $\alpha_{ij} > -1$, $p_j \in \Sigma$ given and

$$A = (a_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

is the $SU(3)$ Cartan matrix.

The Toda system is widely studied in both geometry (description of holomorphic curves in \mathbb{CP}^N , see e.g. [4, 6, 8]) and mathematical physics (non-abelian Chern-Simons vortices theory, see [10, 18, 19]).

In the regular case, Jost, Lin and Wang [11] proved the following important mass-quantization result for sequences of solutions of (1).

Theorem 1.1. Suppose $\alpha_{ij} = 0$ for any i, j and let $u_n = (u_{1,n}, u_{2,n})$ be a sequence of solutions of (1) with $\rho_i = \rho_{i,n}$. Define, for $x \in \Sigma$, $\sigma_1(x), \sigma_2(x)$ as

$$\sigma_i(x) := \lim_{r \rightarrow 0} \lim_{n \rightarrow +\infty} \rho_{i,n} \frac{\int_{B_r(x)} V_i e^{u_{i,n}} dv_g}{\int_{\Sigma} V_i e^{u_{i,n}} dv_g}. \quad (2)$$

Then,

$$(\sigma_1(x), \sigma_2(x)) \in \{(0, 0), (0, 4\pi), (4\pi, 0), (4\pi, 8\pi), (8\pi, 4\pi), (8\pi, 8\pi)\}. \quad (3)$$

In the same paper, the authors state that Theorem 1.1 immediately implies the following compactness result.

Theorem 1.2. Suppose $\alpha_{ij} = 0$ for any i, j and let K_1, K_2 be compact subsets of $\mathbb{R}^+ \setminus 4\pi\mathbb{N}$. Then, the space of solutions of (1) with $\rho_i \in K_i$ satisfying $\int_{\Sigma} u_i dv_g = 0$ is compact in $H^1(\Sigma)$.

Theorem 1.2 is a necessary step to find solutions of (1) by variational methods, as was done in [2, 16, 17].

Although Theorem 1.2 has been widely used, it was not explicitly proved how it follows from Theorem 1.1. Recently, in [13], a proof was given in the case $\rho_1 < 8\pi$.

The purpose of this note is to give a complete proof of Theorem 1.2, extending it to the singular case as well. Actually, the proof follows quite directly from [7].

In the presence of singularities, that is when we allow the α_{ij} to be non-zero, it is convenient to write the system (1) in an equivalent form through the following change of variables:

$$u_i \rightarrow u_i + 4\pi \sum_{j=1}^l \alpha_{ij} G_{p_j} \quad \text{where } G_p \text{ solves } \begin{cases} -\Delta G_p = \delta_p - \frac{1}{|\Sigma|} \\ \int_{\Sigma} G_p dv_g = 0 \end{cases}.$$

The new u_i 's solve

$$-\Delta u_i = \sum_{j=1}^2 a_{ij} \rho_j \left(\frac{\tilde{V}_j e^{u_j}}{\int_{\Sigma} \tilde{V}_j e^{u_j} dv_g} - \frac{1}{|\Sigma|} \right) \quad i = 1, 2. \quad (4)$$

with

$$\tilde{V}_i = \prod_{j=1}^l e^{-4\pi\alpha_{ij} G_{p_j}} V_i \quad \Rightarrow \quad \tilde{V}_i \sim d(\cdot, p_j)^{2\alpha_{ij}} \quad \text{near } p_j.$$

In this case, we still have an analogue of Theorem 1.1 for the newly defined u_i . The finiteness of the local blow-up values has been proved in [14].

We will also show how this quantization result implies compactness of solutions outside a closed, zero-measure set of \mathbb{R}^{+2} .

Theorem 1.3. *There exist two discrete subset $\Lambda_1, \Lambda_2 \subset \mathbb{R}^+$, depending only on the α_{ij} 's, such that for any $K_i \subseteq \mathbb{R}^+ \setminus \Lambda_i$, the space of solutions of (1) with $\rho_i \in K_i$ satisfying $\int_{\Sigma} u_i dv_g = 0$ is compact in $H^1(\Sigma)$.*

As in the regular case, Theorem 1.3 has an important application in the variational analysis of (1), see for instance [2, 1].

2 Proof of the main results

Let us consider a sequence u_n of solutions of (1) with $\rho_i = \rho_{i,n} \xrightarrow{n \rightarrow +\infty} \bar{\rho}_i$ and let us define

$$w_{i,n} := u_{i,n} - \log \int_{\Sigma} \tilde{V}_i e^{u_{i,n}} dv_g + \log \rho_{i,n}, \quad (5)$$

which solves

$$\begin{cases} -\Delta w_{i,n} = \sum_{j=1}^2 a_{ij} \left(\tilde{V}_j e^{w_{j,n}} - \frac{\rho_{j,n}}{|\Sigma|} \right); \\ \int_{\Sigma} \tilde{V}_i e^{w_{i,n}} dv_g = \rho_{i,n} \end{cases} \quad (6)$$

moreover,

$$\sigma_i(x) = \lim_{r \rightarrow 0} \lim_{n \rightarrow +\infty} \int_{B_r(x)} \tilde{V}_i e^{w_{i,n}} dv_g.$$

Let us denote by S_i the blow-up set of $w_{i,n}$:

$$S_i := \left\{ x \in \Sigma : \exists \{x_n\} \subset \Sigma, w_{i,n}(x_n) \xrightarrow{n \rightarrow +\infty} +\infty \right\}.$$

For $w_{i,n}$ we have a concentration-compactness result from [15, 3]:

Theorem 2.1. *Up to subsequences, one of the following alternatives holds:*

- (Compactness) $w_{i,n}$ is bounded in $L^\infty(\Sigma)$ for $i = 1, 2$.

- (Blow-up) The blow-up set $S := S_1 \cup S_2$ is non-empty and finite and $\forall i \in \{1, 2\}$ either $w_{i,n}$ is bounded in $L_{loc}^\infty(\Sigma \setminus S)$ or $w_{i,n} \rightarrow -\infty$ locally uniformly in $\Sigma \setminus S$.
In addition, if $S_i \setminus (S_1 \cap S_2) \neq \emptyset$, then $w_{i,n} \rightarrow -\infty$ locally uniformly in $\Sigma \setminus S$.

Moreover, denoting by μ_i the weak limit of the sequence of measures $\tilde{V}_i e^{w_{i,n}}$, one has

$$\mu_i = r_i + \sum_{x \in S_i} \sigma_i(x) \delta_x$$

with $r_i \in L^1(\Sigma) \cap L_{loc}^\infty(\Sigma \setminus S_i)$ and $\sigma_i(x) \geq 2\pi \min\{1, 1 + \alpha_i(x)\} \forall x \in S_i$, $i = 1, 2$, where

$$\alpha_i(x) = \begin{cases} 0 & \text{if } x \neq p_j \text{ } j = 1, \dots, l \\ \alpha_{ij} & \text{if } x = p_j. \end{cases}$$

Here we want to show that one has $r_i \equiv 0$ for at least one $i \in \{1, 2\}$. It may actually occur that only one of the r_i 's is zero, as shown in [9]. Anyway, to prove Theorems 1.2 and 1.3 we only need one between r_1 and r_2 to be identically zero.

As a first thing, we can show that the profile near blow-up points resembles a combination of Green's functions:

Lemma 2.1. $w_{i,n} - \bar{w}_{i,n} \rightarrow \sum_{j=1}^2 \sum_{x \in S_j} a_{ij} \sigma_j(x) G_{x+s_i}$ in $L_{loc}^\infty(\Sigma \setminus S)$ and weakly in $W^{1,q}(\Sigma)$ for any $q \in (1, 2)$ with $e^{s_i} \in L^p(\Sigma) \forall p \geq 1$.

Proof. If $q \in (1, 2)$

$$\int_{\Sigma} \nabla w_{i,n} \cdot \nabla \varphi dv_g \leq \|\Delta w_{i,n}\|_{L^1(\Sigma)} \|\varphi\|_{\infty} \leq C \|\varphi\|_{W^{1,q'}(\Sigma)}$$

$\forall \varphi \in W^{1,q'}(\Sigma)$ with $\int_{\Sigma} \varphi = 0$, hence one has $\|\nabla w_{i,n}\|_{L^q(\Sigma)} \leq C$. In particular $w_{i,n} - \bar{w}_{i,n}$ converges to a function $w_i \in W^{1,q}(\Sigma)$ weakly in $W^{1,q}(\Sigma) \forall q \in (1, 2)$ and, thanks to standard elliptic estimates, we get convergence in $L_{loc}^\infty(\Sigma \setminus S)$.

The limit functions w_i are distributional solutions of

$$-\Delta w_i = \sum_{j=1}^2 a_{ij} \left(r_j + \sum_{x \in S_j} \sigma_j(x) \delta_x - \frac{\bar{\rho}_j}{|\Sigma|} \right).$$

In particular $s_i := w_i - \sum_{j=1}^2 \sum_{x \in S_j} a_{ij} \sigma_j(x) G_x$ solves

$$-\Delta s_i = \sum_{j=1}^2 a_{ij} \left(r_j + \frac{1}{|\Sigma|} \sum_{x \in S_j} \sigma_j(x) - \frac{\bar{\rho}_j}{|\Sigma|} \right).$$

Since $-\Delta s_i \in L^1(\Sigma)$ we can exploit Remark 2 in [5] to prove that $e^{s_i} \in L^p(\Sigma) \forall p \geq 1$. \square

The following Lemma shows the main difference between the case of vanishing and non-vanishing residual.

Lemma 2.2.

- $r_i \equiv 0 \implies \bar{w}_{i,n} \longrightarrow -\infty$.
- $r_i \not\equiv 0 \implies \bar{w}_{i,n}$ is bounded.

Proof. First of all, $\bar{w}_{i,n}$ is bounded from above due to Jensen's inequality.

Now, take any non-empty open set $\Omega \Subset \Sigma \setminus S$.

$$\int_{\Omega} \tilde{V}_i e^{w_{i,n}} dv_g = e^{\bar{w}_{i,n}} \int_{\Omega} \tilde{V}_i e^{w_{i,n} - \bar{w}_{i,n}} dv_g$$

and by Lemma 2.1

$$\int_{\Omega} \tilde{V}_i e^{w_{i,n} - \bar{w}_{i,n}} dv_g \xrightarrow{n \rightarrow +\infty} \int_{\Omega} \tilde{V}_i e^{\sum_{j=1}^2 \sum_{x \in S_j} a_{ij} \sigma_j(x) G_x + s_i} dv_g \in (0, +\infty).$$

On the other hand,

$$\int_{\Omega} \tilde{V}_i e^{w_{i,n}} dv_g \xrightarrow{n \rightarrow +\infty} \mu_i(\Omega) = \int_{\Omega} r_i(x) dv_g(x).$$

If $r_i \equiv 0$ one has $\bar{w}_{i,n} \longrightarrow -\infty$. If instead $r_i \not\equiv 0$, choosing Ω such that $\int_{\Omega} r_i(x) dv_g > 0$ we must have $\bar{w}_{i,n}$ necessarily bounded. \square

Remark 2.1. From the previous two lemmas, we can write $r_i = \widehat{V}_i e^{s_i}$, where

$$\widehat{V}_i := \tilde{V}_i e^{\lim_{n \rightarrow +\infty} \bar{w}_{i,n} + \sum_{j=1}^2 \sum_{x \in S_j} a_{ij} \sigma_j(x) G_x}$$

satisfies $\widehat{V}_i \sim d(\cdot, x)^{2\alpha_i(x) - \frac{\sum_{j=1}^2 a_{ij} \sigma_j(x)}{2\pi}}$ around each $x \in S_i$, provided $r_i \not\equiv 0$.

The key Lemma is an extension of Chae-Ohtsuka-Suzuki [7] to the singular case. Basically, it gives necessary conditions on the σ_i 's to have non-vanishing residual.

Lemma 2.3. *For both $i = 1, 2$ we have $s_i \in W^{2,p}(\Sigma)$ for some $p > 1$. Moreover, if $\sum_{j=1}^2 a_{ij}\sigma_j(x_0) \geq 4\pi(1 + \alpha_i(x_0))$ for some $x_0 \in S_i$, then $r_i \equiv 0$.*

Proof. If both r_1 and r_2 are identically zero, then also s_1 and s_2 are both identically zero, so there is nothing to prove.

Suppose now $r_1 \not\equiv 0$ and $r_2 \equiv 0$. In this case,

$$\begin{cases} -\Delta s_1 = 2 \left(r_1 + \frac{1}{|\Sigma|} \sum_{x_0 \in S_1} \sigma_1(x_0) - \frac{\bar{p}_1}{|\Sigma|} \right) \\ -\Delta s_2 = - \left(r_1 + \frac{1}{|\Sigma|} \sum_{x_0 \in S_1} \sigma_1(x_0) - \frac{\bar{p}_1}{|\Sigma|} \right) \end{cases}.$$

Then, being $G_x(y) \geq -C$ for all $x, y \in \Sigma$ with $x \neq y$, we get

$$s_1(x) = \int_{\Sigma} G_x(y) 2r_1(y) dv_g(y) \geq -2C \int_{\Sigma} r_1 dv_g \geq -C'.$$

Therefore, from the previous remark, around each $x_0 \in S_1$ we get

$$r_1(y) \geq C d(x_0, y)^{2\alpha_1(x_0) - \frac{\sum_{j=1}^2 a_{1j}\sigma_j(x_0)}{2\pi}},$$

so being $r_1 \in L^1(\Sigma)$, it must be $\sum_{j=1}^2 a_{1j}\sigma_j(x_0) < 4\pi(1 + \alpha_1(x_0))$.

Moreover, being $e^{qs_1} \in L^1(\Sigma)$ for any $q \geq 1$, from Holder's inequality we get $r_1 \in L^p(\Sigma)$ for some $p > 1$; therefore, standard estimates yield $s_i \in W^{2,p}(\Sigma)$ for both $i = 1, 2$.

Consider now the case of both non-vanishing residuals, which means by Theorem 2.1 $S_1 = S_2 = S$. In this case,

$$-\Delta \left(\frac{2s_1 + s_2}{3} \right) = \left(r_1 + \frac{1}{|\Sigma|} \sum_{x_0 \in S_1} \sigma_1(x_0) - \frac{\bar{p}_1}{|\Sigma|} \right)$$

hence, arguing as before, $\frac{2s_1 + s_2}{3} \geq -C$. Therefore, using the convexity of $t \rightarrow e^t$ we get

$$\begin{aligned} C \int_{\Sigma} \min \{ \widehat{V}_1, \widehat{V}_2 \} dv_g &\leq \int_{\Sigma} \min \{ \widehat{V}_1, \widehat{V}_2 \} e^{\frac{2s_1 + s_2}{3}} dv_g \leq \\ &\leq \frac{2}{3} \int_{\Sigma} \widehat{V}_1 e^{s_1} dv_g + \frac{1}{3} \int_{\Sigma} \widehat{V}_2 e^{s_2} dv_g = \frac{2}{3} \int_{\Sigma} r_1 dv_g + \frac{1}{3} \int_{\Sigma} r_2 dv_g < +\infty. \end{aligned}$$

Therefore, for any $x_0 \in S$ there exists $i \in \{1, 2\}$ such that $\sum_{j=1}^2 a_{ij}\sigma_j(x_0) < 4\pi(1 + \alpha_i(x_0))$. Fix x_0 and suppose, without loss of generality, that this is true for $i = 1$. This implies that $r_1 \in L^p(B_r(x_0))$ for small r , so for $x \in B_{\frac{r}{2}}(x_0)$ we have

$$\begin{aligned}
s_2(x) &= \int_{\Sigma} G_x(y) 2r_2(y) dv_g(y) - \int_{B_r(x_0)} G_x(y) r_1(y) dv_g(y) \\
&\quad - \int_{\Sigma \setminus B_r(x_0)} G_x(y) r_1(y) dv_g(y) \\
&\geq -C - \sup_{z \in \Sigma} \|G_z\|_{L^{p'}(\Sigma)} \|r_1\|_{L^p(B_r(x_0))} \\
&\quad - \sup_{z \in B_{\frac{r}{2}}(x_0)} \|G_z\|_{L^\infty(\Sigma \setminus B_r(x_0))} \|r_1\|_{L^1(\Sigma)} \\
&\geq -C'.
\end{aligned}$$

Therefore, arguing as before, we must have $\sum_{j=1}^2 a_{2j}\sigma_j(x_0) < 4\pi(1 + \alpha_2(x_0))$ and $r_2 \in L^p(B_{\frac{r}{2}}(x_0))$. This implies $-\Delta s_i \in L^p(B_{\frac{r}{2}}(x_0))$ for both i 's. Hence, being x_0 arbitrary and $-\Delta s_i \in L^p_{loc}(\Sigma \setminus S)$, by elliptic estimates the proof is complete. \square

From Lemmas 2.1 and 2.3 we can deduce, through a Pohozaev identity, the following information about the local blow-up values. This was explicitly done in [12, 14].

Lemma 2.4. *If $x_0 \in S$ then*

$$\sigma_1^2(x_0) + \sigma_2^2(x_0) - \sigma_1(x_0)\sigma_2(x_0) = 4\pi(1 + \alpha_1(x_0))\sigma_1(x_0) + 4\pi(1 + \alpha_2(x_0))\sigma_2(x_0).$$

Lemma 2.5. *If $x_0 \in S_1 \cap S_2$ then there exists i such that $\sum_{j=1}^2 a_{ij}\sigma_j(x_0) \geq 4\pi(1 + \alpha_i(x_0))$.*

Proof. Suppose the statement is not true. Then, by Lemmas 2.3 and 2.4, we would have

$$\begin{cases} 2\sigma_1(x_0) - \sigma_2(x_0) < 4\pi(1 + \alpha_1(x_0)) \\ 2\sigma_2(x_0) - \sigma_1(x_0) < 4\pi(1 + \alpha_2(x_0)) \\ \sigma_1^2(x_0) + \sigma_2^2(x_0) - \sigma_1(x_0)\sigma_2(x_0) = \\ = 4\pi(1 + \alpha_1(x_0))\sigma_1(x_0) + 4\pi(1 + \alpha_2(x_0))\sigma_2(x_0) \end{cases}, \quad (7)$$

which has no solution between positive $\sigma_1(x_0), \sigma_2(x_0)$.

In fact, by multiplying the first equation by $\frac{\sigma_1(x_0)}{2}$ and the second by

$\frac{\sigma_2(x_0)}{2}$ and summing, we get

$$\sigma_1^2(x_0) + \sigma_2^2(x_0) - \sigma_1(x_0)\sigma_2(x_0) < 2\pi(1+\alpha_1(x_0))\sigma_1(x_0) + 2\pi(1+\alpha_2(x_0))\sigma_2(x_0),$$

which contradicts the third equation.

The scenario is described by the picture.

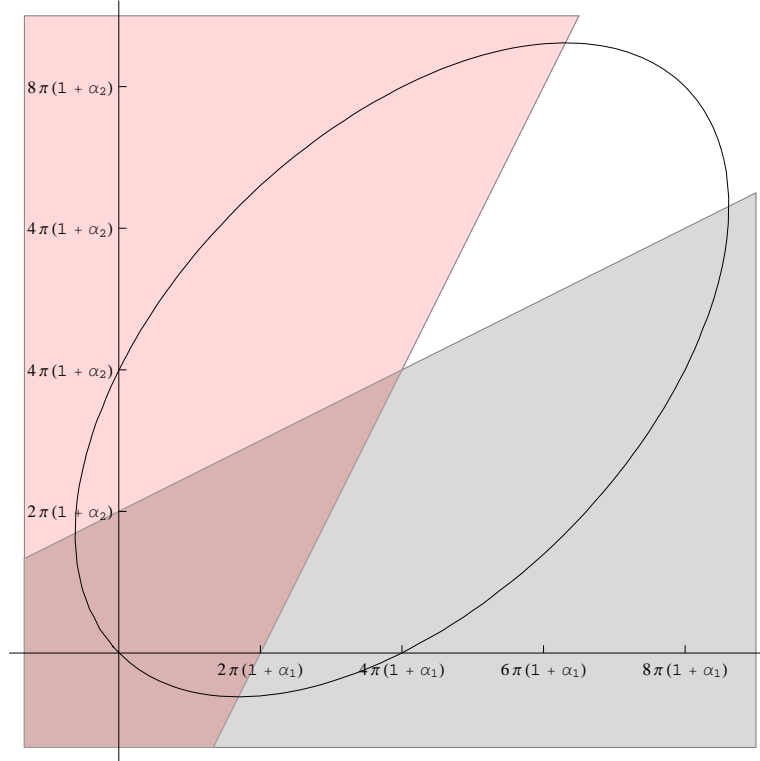


Figure 1: The algebraic conditions (7) satisfied by $\sigma_1(x_0), \sigma_2(x_0)$

□

Corollary 2.1. *Let w_n be a sequence of solutions of (6). If $S \neq \emptyset$ then either $r_1 \equiv 0$ or $r_2 \equiv 0$. In particular there exists $i \in \{1, 2\}$ such that $\bar{\rho}_i = \sum_{x \in S_i} \sigma_i(x)$.*

Proof of Theorems 1.2 and 1.3.

Let u_n be a sequence of solutions of (1) with $\rho_i = \rho_{i,n} \xrightarrow{n \rightarrow +\infty} \bar{\rho}_i$ and

$$\int_{\Sigma} u_{1,n} dv_g = \int_{\Sigma} u_{2,n} dv_g = 0 \text{ and let } w_{i,n} \text{ be defined by (5).}$$

If both $w_{1,n}$ and $w_{2,n}$ are bounded from above, then by standard estimates u_n is bounded in $W^{2,p}(\Sigma)$, hence is compact in $H^1(\Sigma)$.

Otherwise, from Corollary 2.1 we must have $\bar{\rho}_i = \sum_{x \in S_i} \sigma_i(x)$ for some $i \in \{1, 2\}$. In the regular case, from Theorem 1.1 follows that ρ_i must be an integer multiple of 4π , hence the proof of Theorem 1.2 is complete.

In the singular case, local blow-up values at regular points are still defined by (3), whereas for any $j = 1, \dots, l$ there exists a finite Γ_j such that $(\sigma_1(p_j), \sigma_2(p_j)) \in \Gamma_j$. Therefore, it must hold

$$\rho_i \in \Lambda_i := \left\{ 4\pi k + \sum_{j=1}^l n_j \sigma_j, \ k \in \mathbb{N}, \ n_j \in \{0, 1\}, \ \sigma_j \in \Pi_i(\Gamma_j) \right\},$$

where Π_i is the projection on the i^{th} component; being Λ_i discrete we can also conclude the proof of Theorem 1.3. \square

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